SELF-OSCILLATORY FLOW IN ASYMPTOTIC BOUNDARY LAYERS

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The loss of stability of a suction-controlled boundary-layer flow with respect to small but finite perturbations is analyzed. Hard excitation of self-oscillations is a special characteristic of this kind of flow. An unstable state of self-oscillation exists within a certain distance of the neutral curve (in the range of stability of the original flow) for all Reynolds numbers Re.

The use of external controlling factors, such as a negative pressure gradient, suction, a transverse magnetic field (in the case of a conducting medium), and so on, enables the laminar mode of flow in a boundary layer to be extended to fairly high Reynolds numbers, this effect being due to the greater degree of occupation of the velocity profiles. It is nevertheless found that this extended laminar state loses stability with respect to finite perturbations in an explosive manner for Reynolds numbers smaller than the critical value calculated from the linear theory.

After the loss of stability the original steady-state flow is replaced by a periodic flow of the uniform alternating self-oscillatory type. The existence of self-oscillations of this kind as solutions of the Navier–Stokes equations branching off from the steady-state solution was proved in [1]. The self-oscillations branch off at Reynolds numbers corresponding to points on the neutral curve of the linear theory of stability. The nonlinear stability of flow in a plane channel was studied in [2], and the "hard" character of the loss of stability experienced by the original flow was proved.

§ 1. For uniform suction at a constant velocity v_0, the asymptotic profile of the boundary layer based on the displacement thickness $\delta_* = \nu/|v_0|$ and the velocity at infinity u_∞ takes the form

$$u = 1 - e^{-y}. \tag{1.1}$$

Here (1.1) is the exact solution of the Navier–Stokes equations for the case of plane-parallel motion [3]. Allowing for the transverse velocity component, the stream function of the perturbed motion satisfies the equation

$$\begin{aligned}\partial\Delta\psi/\partial t + u\partial\Delta\psi/\partial x - u''\partial\psi/\partial x - (1/\mathrm{Re})\Delta\Delta\psi - (1/\mathrm{Re})\partial\Delta\psi/\partial y = \\ = (\partial\psi/\partial x)\partial\Delta\psi/\partial y - (\partial\psi/\partial y)\partial\Delta\psi/\partial x\end{aligned} \tag{1.2}$$

with boundary conditions based on the attachment of the flow to the wall and the requirement of minimum growth as $y \to \infty$. Here $\mathrm{Re} = u_\infty \delta_*/\nu$ is based on the displacement thickness of the boundary layer δ_*; Δ is the Laplace operator.

Following [2], we take an arbitrary point corresponding to Re_0 on the neutral curve of the linearized Eq. (1.2). We put

$$\mathrm{Re} = \mathrm{Re}_0 + \varepsilon^2 f,\ f = \pm 1 \tag{1.3}$$

and seek the solution to (1.2) in the form of a series in terms of the small parameter ϵ

$$\psi = \frac{1}{\mathrm{Re}} \sum_{k=1}^{\infty} \varepsilon^k \psi_k (x - ct, y),\ \mathrm{Re}c = \mathrm{Re}_0 \sum_{k=0}^{\infty} c_k \varepsilon^k. \tag{1.4}$$

Substituting (1.4) into (1.2) and equating the coefficients of ϵ^k, to zero, we obtain a series of equations,

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$$\Delta\Delta\psi_k - \mathrm{Re}_0\left[(u-c_0)\frac{\partial\Delta\psi_k}{\partial x} - u''\frac{\partial\psi_k}{\partial x}\right] + \frac{\partial\Delta\psi_k}{\partial y} =$$

$$= f\left[u\frac{\partial\Delta\psi_{k-2}}{\partial x} - u''\frac{\partial\psi_{k-2}}{\partial x}\right] + \sum_{j=1}^{k-1}\left[\left(\frac{\partial\psi_{k-j}}{\partial y} - \mathrm{Re}_0 c_{k-j}\right)\frac{\partial\Delta\psi_j}{\partial x} - \frac{\partial\psi_{k-j}}{\partial x}\frac{\partial\Delta\psi_j}{\partial y}\right]. \tag{1.5}$$

For k = 1, Eq. (1.5) is homogeneous and its solution takes the form

$$\psi_1(x-ct, y) = \beta\{\varphi(y)\exp[\,i\alpha(x-ct)] + \bar{\varphi}(y)\exp[\,-i\alpha(x-ct)]\},$$

where $\varphi(y)$ is the solution to the modified Orr–Sommerfeld equation,

$$L_\alpha\varphi \equiv \varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi - i\alpha\,\mathrm{Re}_0[(u-c_0)(\varphi''-\alpha^2\varphi) - u''\varphi] + (\varphi''' - \alpha^2\varphi') = 0 \tag{1.6}$$

with boundary conditions $\varphi = \varphi' = 0$ at $y = 0$ and the requirement of minimum growth in $\varphi(y)$ at infinity. For sufficiently large y we may take $u'' = 0$, $u = 1$. The solutions of (1.6) falling off at infinity then take the form

$$\varphi = C_1\exp(-\alpha y) + C_2\exp(-\gamma y),$$

where

$$\gamma = \gamma_r + i\gamma_i;\ \gamma_r = 1/2 + \varkappa_r;\ \varkappa_r = (\sqrt{a + \sqrt{(a^2+b^2)}}/2;$$
$$\gamma_i = b/2\varkappa_r;\ a = \alpha^2 + 1/4;\ b = \alpha\,\mathrm{Re}_0(1-c_0).$$

Allowing for this kind of attenuation in the perturbations, the boundary conditions at the outer boundary (for reasonably large y) may be written

$$(\varphi'' - \alpha^2\varphi)' + \gamma(\varphi'' - \alpha^2\varphi) = 0;$$
$$(\varphi'' - \gamma^2\varphi)' + \alpha(\varphi'' - \gamma^2\varphi) = 0,\ y = A.$$

Numerical calculations were carried out for a finite range of values $0 < y < A$, where $A = 10$. Increasing the range of integration had hardly any effect on the results.

In expansion (1.4) for the stream function, the term corresponding to $k = 2$ takes the form

$$\psi_2 = \beta^2\{V_0(y) + V_1(y)\exp[2i\alpha(x-ct)] + \bar{V}_1(y)\exp[-2i\alpha(x-ct)]\}.$$

The function $V_1(y)$ should satisfy the inhomogeneous equation

$$L_{2\alpha}V_1 = i\alpha(\varphi'^2 - \varphi\varphi'')'$$

with boundary conditions $V_1 = V_1' = 0$ at $y = 0$, and with the minimum growth condition at infinity. The function $V_0(y)$ satisfies the boundary problem

$$V_0^{IV} + V_0'' = i\alpha(\bar{\varphi}\varphi' - \varphi\bar{\varphi}')'',\ V_0 = V_0' = 0 \quad \text{at} \quad y = 0$$

and the condition of minimum $V_0(y)$ growth at infinity, which may be expressed in the form

$$\begin{cases} V_0' - V_0''' = 0; \\ V_0'' + V_0''' = 0 \quad \text{at} \quad y = A. \end{cases}$$

The condition for the solubility of (1.5) when $k > 1$ is the equation $c_k = 0$ for odd k.

It follows from (1.5) that ψ_3 will be the sum of harmonic oscillations with wave numbers α and 3α. The condition for the solubility of the equation governing the amplitude of the first harmonic takes the form

$$-c_2\,\mathrm{Re}_0 I_1 + \beta^2 I_2 + f I_3 = 0, \tag{1.7}$$

$$\text{where } I_1 = \int_0^\infty \theta(y)(\varphi'' - \alpha^2\varphi)\,dy;$$

$$I_2 = \int_0^\infty \theta(y)\,[V_0'(\varphi'' - \alpha^2\varphi) - V_1'(\bar{\varphi}'' - \alpha^2\bar{\varphi}) - 2V_1(\bar{\varphi}''' - \alpha^2\bar{\varphi}') -$$
$$- \varphi V_0''' + 2\bar{\varphi}'(V_1'' - 4\alpha^2 V_1) + \bar{\varphi}(V_1''' - 4\alpha^2 V_1')]\,dy;$$

$$I_3 = \int_0^\infty \theta(y)\,[u(\varphi'' - \alpha^2\varphi) - u''\varphi]\,dy.$$

Here $\theta(y)$ is a solution of the problem conjugate to (1.6). Integration is carried out with respect to an infinite region, but the integrals converge by virtue of the exponential attenuation of the integrands. In view of the fact that on the neutral curve

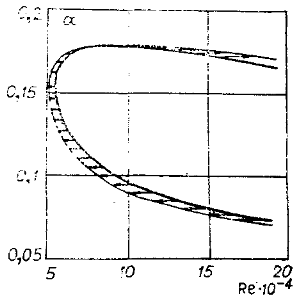

Fig. 1.

$$\frac{\partial E}{\partial \mathrm{Re}} = f\beta^2 \int_0^\infty (|\varphi'|^2 + \alpha^2|\varphi|^2)\,dy\,,$$

where E is the energy of the self-oscillations, the sign of f in (1.7) coincides with the sign of the derivative $\partial E/\partial \mathrm{Re}$ on condition that $\beta^2 > 0$. This coincidence may be interpreted as the existence of supercritical or subcritical self-oscillations [see (1.3)].

Numerical calculations carried out by the method of differential advance and matching of the solutions [4] showed that in the case under consideration $\partial E/\partial \mathrm{Re} < 0$ on the whole of the lower branch of the neutral curve and part of the upper branch, i.e., the self-oscillations branched in the direction of smaller Reynolds numbers. Self-oscillations exist in the region corresponding to the stability of the original flow, but are themselves unstable. The foregoing results are illustrated in Fig. 1, which indicates the neutral curve for asymptotic flow in a boundary layer with suction. The arrows show where the self-oscillatory conditions exist; their direction corresponds to the sign of the derivative $\partial E/\partial \mathrm{Re}$. The change in the sign of $\partial E/\partial \mathrm{Re}$, which passes through zero at the point $\alpha = 0.1807$, Re = 87612, does not have any fundamental significance, but is associated with a change in the orientation of the stability range relative to the neutral curve. The character of the branching at the tip of the neutral curve, at which $\alpha_* = 0.1557$, $\mathrm{Re}_* = 54370$, is of fundamental significance. If the self-oscillations branch into the region of stability, then for $\mathrm{Re} < \mathrm{Re}_*$ the original flow is unstable with respect to finite perturbations, and self-oscillations of finite amplitude or turbulence may arise instantaneously for the first time. Thus, the nonlinear critical Reynolds number is smaller than that calculated by the linear theory. The same results as to the character of the branching are obtained if, in Eq. (1.2), we omit the term responsible for the existence of a transverse rate of suction, but there is a slight change in the critical parameters, which are now $\mathrm{Re}_* = 47100$, $\alpha_* = 0.1625$.

§2. The velocity profile of the boundary (wall) flow of a conducting, incompressible liquid subject to a fairly strong transverse magnetic field takes the same form (1.1), the only difference being that the displacement thickness $\delta_* = 1/G$, where G is the Hartmann number. In Eq. (1.2) instead of the term associated with the constant transverse velocity, i.e., $(1/\mathrm{Re})\partial\Delta\psi/\partial y$, we shall have an additional term of the form $(1/\mathrm{Re}_*)\cdot \partial^2\psi/\partial y^2$. The parameters corresponding to the tip of the neutral curve will be $\mathrm{Re}_* = 48{,}300$, $\alpha_* = 0.1617$. Otherwise, the branching of the steady-state solutions will be completely analogous to that described in the foregoing. On the whole neutral curve self-oscillations will branch into the region corresponding to the stability of the original flow. At the tip of the neutral curve $\partial E/\partial \mathrm{Re} < 0$, i.e., self-oscillations will exist for Reynolds numbers smaller than the critical value given by the linear theory. The calculations were in every case continued to Re values greater than those shown in Fig. 1.

There is a well-known result according to which the stability loss bears a soft character for Blasius flow, but is of a hard character for even slight negative pressure gradients, although a negative pressure gradient increases the stability of the flow in the boundary layer relatively to infinitely small perturbations.

Thus, by making use of appropriate control factors we may extend the laminar mode of flow to very large values of Re. At the same time, perturbations of finite amplitude lead to a loss of stability of the original steady-state process for Reynolds numbers considerably smaller than those indicated by the linear theory. This implies that the nonlinear critical Re number possesses conservative properties with respect to the external controlling factors.

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